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Flexural vibrations of a moving rod☆

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ABSTRACT

The problem of the transverse natural vibrations of part of a rod between two coaxially fixed guides, moving with an arbitrary constant velocity, is investigated. The conditions of rigid clamping are taken as the boundary conditions. Additional shear stresses, due to longitudinal tension or compression are taken into account. Relations defining the natural frequencies and forms are constructed in an exact formulation by Fourier method. The dependence of the natural frequencies and forms of the lowest vibration modes on the rate of displacement, unknown in the literature, are constructed, and their features are established. A modelling and animation of unusual wave motions of the rod are presented. The main characteristics for the higher vibration modes are constructed.

1. Formulation of the problem

We will investigate the plane transverse vibrations of a uniformly moving part of a rod between two fixed clamps (see Fig. 1). The rod is assumed to be thin and uniform; it can be subjected to a longitudinal stress – tension or compression. These are assumed to be moderate, which enables longitudinal deformations to be neglected.

Note that one arrives at a similar formulation of the problem by considering the transverse vibrations of a pipeline, containing a moving liquid.[1] If we assume that the liquid mass per unit length is considerably greater than the mass per unit length of the pipeline material, the boundary-value problem allows of an exact analytical solution to be obtained (see below).

We will construct a mathematical model of small transverse vibrations of a "moving" rod corresponding to these assumptions.[1,2]

Flexural vibrations in a moving system of coordinates are described in the linear approximation by an equation of state

$$\rho\frac{D^2u}{Dt^2} - TS\frac{\partial^2 u}{\partial x_*^2} + EI\frac{\partial^4 u}{\partial x_*^4} = 0 \tag{1.1}$$

Here x_* is the Lagrangian coordinate, $0 \le x_* \le l$, ρ is the linear density of the rod, TS is the tensile stress ($T > 0$) or compressive strength ($T < 0$), E is Young's modulus, S is the area, I is the moment of inertia of the cross section and l is the distance between the clamps, i.e., the length of the vibrating part of the rod.

Changing to a fixed (Eulerian) coordinate x^*, the expressions for the substantional derivatives have the form

$$\frac{Du}{Dt} = \dot{u} + \upsilon_0 u', \quad \frac{D^2u}{Dt^2} = \ddot{u} + 2\upsilon_0\dot{u}' + \upsilon_0^2 u'' \tag{1.2}$$

Here υ_0 is the longitudinal rate of displacement of points of the neutral line of the rod, the dot denotes a derivative with respect to the time t, and a prime denotes a derivative with respect to the Eulerian coordinate x^*.

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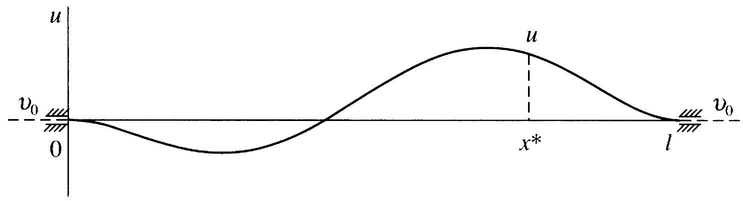

Fig. 1.

After substituting the second derivative of (1.2) into Eq. (1.1) it is reduced to the more usual form. Reducing the number of parameters by dividing the equation by ρ and introducing a dimensionless coordinate, we obtain

$$\ddot{u} + 2\upsilon\dot{u}' + \upsilon^2 u'' - c^2 u'' + \nu^2 u^{IV} = 0, \quad 0 \le x \le 1$$
$$x = \frac{x^*}{l}, \quad \upsilon = \frac{\upsilon_0}{l}, \quad c^2 = \frac{TS}{\rho l^2} \quad (T > 0), \quad \nu^2 = \frac{EI}{\rho l^4} \tag{1.3}$$

The parameters υ, c and ν introduced have the dimension of frequency (angular velocity). When $T < 0$ (compression) the term $-c^2$ changes to $+c^2$; the case when $T > 0$ (1.3) is of considerable interest for applications.

To describe the natural vibrations of a rod moving between clamps at the points $x = 0$, $x = 1$, we impose the appropriate boundary conditions

$$u(t, 0) = u'(t, 0) = u(t, 1) = u'(t, 1) = 0, \quad t \ge 0 \tag{1.4}$$

The initial conditions for system (1.3) and (1.4) are not specified. The problem is to find the natural vibrations and to analyse the conditions for them to exist. It is required to determine and to investigate the natural frequencies and vibration forms, which is a non-trivial problem.

We will seek a solution in the form of the complex expression (Fourier method)

$$u(t, x) = U(x)\exp i\omega t$$

where ω is a real parameter (the vibration frequency), and U is a complex function, which is to be determined. We obtain the following generalized boundary-value problem for finding ω and U

$$-\omega^2 U + 2i\omega\upsilon U' - (c^2 - \upsilon^2)U'' + \nu^2 U^{IV} = 0; \quad x = 0, 1: U = U' = 0 \tag{1.5}$$

which, as can be established by integration by parts, is non-self-conjugate.[3,4] The conjugate of Eq. (1.5) is the equation with the minus sign before the imaginary unit. Hence, the presence of a displacement ($\upsilon \ne 0$) of the rod leads to a non-self-conjugacy effect, that corresponding complicates the investigation of the problem. In particular, different solutions of problem (1.5) are not orthogonal, which can be established directly using standard methods (see below).

To construct a transcendental characteristic equation in ω based on the general solution, we will represent the required function $U(x)$ in the form of an exponential function. We obtain the auxiliary relation

$$U = C\exp kx, \quad \nu^2 k^4 - (c^2 - \upsilon^2)k^2 + 2i\omega\upsilon k - \omega^2 = 0 \tag{1.6}$$

from which, by Ferrari formulae, the complex roots $k_1, \ldots, k_4$ can be calculated as algebraic functions in the form of radicals of unknown frequency ω and the specified parameters ν, c and υ. Taking boundary conditions (1.5) into account, we obtain the required expression for determining $\Delta(\omega)$, the zeros of which determine the natural frequencies ω_n of the vibration of the moving rod. In fact, by a standard procedure, we can represent the determinant in the following compact form

$$\Delta(\omega) = k_{21}k_{41}p_{31}p_{42} - k_{31}p_{21}(k_{21}p_{32} + k_{41}p_{43})$$
$$k_{\sigma\kappa} = k_\sigma - k_\kappa, \quad p_{\sigma\kappa} = p_\sigma - p_\kappa; \quad \sigma, \kappa = 1, \ldots, 4$$
$$k_\sigma = k_\sigma(\omega), \quad p_\sigma = p_\sigma(\omega) \equiv \exp k_\sigma(\omega); \quad \omega_n = \mathrm{Arg}_\omega \Delta(\omega), \quad n = 1, 2, \ldots \tag{1.7}$$

According to the general procedure, it is required to obtain the real zeros ω_n of the function $\Delta(\omega)$ for specified values of the parameters ν, c and υ, which can be achieved by numerical methods. The complex eigenfunctions $U_n(x)$ corresponding to the n-th mode are found in a standard way in the form

$$U_n(x) = C_n \sum_{\sigma=1}^{4} a_\sigma(\omega_n)\exp k_\sigma(\omega_n)x, \quad n = 1, 2, \ldots \tag{1.8}$$

Here C_n, α_σ, k_σ are complex coefficients, and the values of a_σ are calculated in terms of the known complex quantities $k_\kappa(\omega_n)$, $p_\kappa(\omega_n)$, while C_n are determined by the initial data or the "normalization" conditions (see below).

The use of the general relations for the natural frequencies $\omega_n(\nu, c, \upsilon)$ and the forms $U_n(x, \nu, c, \upsilon)$, constructed from relations (1.5)–(1.8), to analyse the effect of all these parameters is extremely time consuming, and hence simplifying assumptions are required. Note that the number of parameters can be reduced by one by appropriate normalization. Normalization by ν or c is meaningful for applications due to the decisive influence of the flexural rigidity of a rod or the tensile force of a string on the vibration properties.

Below we investigate the solutions of problem (1.5) under simplifying conditions, in which one of the parameters $|\upsilon|$, ν or c is assumed to be small, in particular, equal to zero. The main attention is devoted to the situations when the tension T in Eq. (1.1) can be neglected, i.e., we assume $c=0$ in relations (1.3), (1.5) and (1.6). This case is of interest in applications; it has hardly been investigated in the literature[1–4] for comparatively high values of $|\upsilon|$.

2. Analysis of the frequencies and forms of the natural vibrations for a relatively low displacement velocity of the rod

It follows from Eq. (1.6) that the effect of a displacement will be weak if $|\upsilon| \ll c$ or $|\upsilon| \ll \nu$. An approximate solution $k(\omega, \nu, c, \upsilon)$ can be constructed by the method of perturbations (using Weierstrass theorem[5]) based on a generating solution, corresponding to the classical self-conjugate problem (1.5) (with $\upsilon = 0$). In fact, we obtain for the roots of the auxiliary relation (1.6), which has the form of a biquadratic equation,

$$k_{1,2}^0 = \pm ip^-(\omega),\quad k_{3,4}^0 = \pm p^+(\omega),\quad 0 \le \omega < \infty;\quad p^{\mp} = \frac{1}{\sqrt{2}\nu}((c^4 + 4\nu^2\omega^2)^{1/2} \mp c^2)^{1/2} \tag{2.1}$$

We will use the common scheme (1.6), (1.7) when deriving the characteristic equation based on the values of $k_\sigma^0(\omega)(\sigma = 1, 2, 3, 4)$ (2.1) obtained. The required real function $U^0(x)$ is expressed by means of trigonometric sines and cosines of p^-x and hyperbolic signs and cosines of p^+x. From boundary conditions (1.5) we obtain expressions for the determinant $\Delta^0(\omega)$ and the natural frequencies ω_n^0 of the vibration of a fixed tensioned rod with clamped ends

$$\Delta^0(\omega) = (p^{-2} - p^{+2})\sin p^- \operatorname{sh} p^+ - 2p^-p^+(1 - \cos p^- \operatorname{ch} p^+) \tag{2.2}$$

$$p^-p^+ = \frac{\omega}{\nu},\ p^{-2} - p^{+2} = -\frac{c^2}{\nu^2};\ p^- = p^+ = \left(\frac{\omega}{\nu}\right)^{1/2},\ c \to 0;\ p^- = \frac{\omega}{c},\ p^+ = \frac{c}{\nu},\ \nu \to 0$$

$$\omega_n^0(\nu, c) = \mathrm{Arg}_\omega \Delta^0(\omega),\quad n = 1, 2, \ldots$$

Note that expression (2.2) for Δ^0 turns out to be real, and may have a denumerable system of roots ω_n^0. The values of the latter can be found numerically and can be analysed as a function of the parameters ν and c and the number of the mode n. In the limiting cases when $\nu = 0$ or $c = 0$ we obtain the well-known expressions[6,7]

$$\nu = 0:\ \sin\frac{\omega}{c} = 0,\quad \omega_n^0 = \pi nc$$

$$c = 0:\ 1 - \cos\sqrt{\frac{\omega^0}{\nu}}\operatorname{ch}\sqrt{\frac{\omega^0}{\nu}} = 0;\quad \sqrt{\frac{\omega_1^0}{\nu}} = 4.73,\quad \sqrt{\frac{\omega_2^0}{\nu}} = 7.86,\quad \sqrt{\frac{\omega_3^0}{\nu}} = 11.00, \ldots$$

$$\sqrt{\frac{\omega_n^0}{\nu}} \approx \left(n + \frac{1}{2}\right)\pi + O(\exp(-\pi n)),\quad n \gg 1 \tag{2.3}$$

which correspond to vibration of a string or a rod.

The natural forms of vibration are found in a standard way and, using (1.8), can be reduced to the form

$$U_n^0(x) = C_n^0\left[\sin p_n^0 x - \left(\frac{p_n^0}{q_n^0}\right)\operatorname{ch} q_n^0 x + p_n^0\frac{\cos p_n^0 - \operatorname{ch} q_n^0}{p_n^0 \sin p_n^0 + q_n^0 \operatorname{sh} q_n^0}(\cos p_n^0 x - \operatorname{ch} q_n^0 x)\right]$$

$$(U_n^0(x), U_m^0(x)) = C_n^0 C_m^0 \delta_{nm},\quad n, m = 1, 2, \ldots,\quad p_n^0 = p^-(\omega_n^0),\quad q_n^0 = p^+(\omega_n^0) \tag{2.4}$$

The functions $U_n^0(x)$ (2.4) possess the properties of orthogonality and form a basis in the space of functions that are square integrable.[6] They have $n-1$ intermediate nodes (zeros) and n extrema. The constants C_n^0 are chosen from additional conditions, in particular, from the initial conditions or the normalization conditions. The vibrations of different points of the rod are synchronous and the nodes are fixed.

We will construct an approximate solution of the problem for small values of the parameter υ. Since the roots k_σ^0 (2.1) are simple, the perturbed values can be represented in the form of expansions in integer powers of υ (Ref. 5)

$$k = k^0(\omega) + \upsilon k^{(1)}(\omega) + \upsilon^2 k^{(2)}(\omega) + \ldots + \upsilon^j k^{(j)}(\omega) + \ldots$$

$$k_{1,2}^{(1)} = -k_{3,4}^{(1)} = i\omega(c^4 + 4\nu^2\omega^2)^{-1/2}$$

$$2\nu^2 k_{1,2}^{0^2} - c^2 = -(2\nu^2 k_{3,4}^{0^2} - c^2) = -(c^4 + 4\nu^2\omega^2)^{1/2}$$

$$k_\sigma^{(2)} = -(2\nu^2 k_\sigma^{0^2} - c^2)^{-1/2}\left(3k_\sigma^0 k_\sigma^{(1)^2} - \frac{c^2 k_\sigma^{(1)^2}}{2k_\sigma^0} + \frac{k_\sigma^0}{2} + \frac{2i\omega k_\sigma^{(1)}}{k_\sigma^0}\right) \tag{2.5}$$

Note that the coefficients $k_\sigma^{(j)}$ are imaginary for odd values of j and all values σ of the power of the expansion in υ; for even values of j the coefficients $k_{1,2}^{(j)}$ are imaginary while $k_{3,4}^{(j)}$ are real. This property will be confirmed later when considering other limiting cases of Eq.

(1.6): when $\nu = 0$ or $c = 0$ for arbitrary admissible values of υ. It indicates that the characteristic determinant for ω leads to a real function (see below) of the form

$$\Delta(\omega) = \Delta^0(\omega) + \upsilon\Delta^{(1)}(\omega) + \upsilon^{(2)}\Delta^{(2)}(\omega) + \ldots + \upsilon^j\Delta^{(j)}(\omega) + \ldots$$
$$\omega_n = Arg_\omega\Delta(\omega) = \omega_n^0 + \upsilon\omega_n^{(1)} + \upsilon^2\omega_n^{(2)} + \ldots\upsilon^j\omega_n^{(j)} + \ldots; \quad j = 1, 2, \ldots \tag{2.6}$$

The expressions for the coefficients $\Delta^{(j)}(\omega)$ (2.6) are extremely lengthy and can be obtained using symbolic calculations. An approximate analytical calculation of the coefficients for the natural frequencies $\omega_n^{(j)}$ and also for the expressions for the eigenfunctions, is even more lengthy and inefficient. It is preferable to use numerical-analytical procedures as they apply to the initial problem (1.5), (1.6). The corresponding approach to calculating the natural frequencies and forms is described in Section 4 for $c = 0$.

Another limiting case of the problem, when ω is relatively large, is of interest. This corresponds to fairly high modes of vibrations ($n \gg 1$). It follows from Eq. (1.6) and the physical considerations, that the effect of the value of υ will be relatively small when $\nu > 0$. In fact, instead of the unknown dimensionless parameter k we will introduce the unknown $h = k\sqrt{\nu/\omega}$. Then

$$h^4 - \frac{c^2 - \upsilon^2}{\nu\omega}h^2 + \frac{2i\upsilon}{\sqrt{\nu\omega}}h - 1 = 0$$
$$h \approx h^0 + \frac{1}{4h^{0^2}}\left(\frac{c^2 - \upsilon^2}{\nu\omega}h^0 - \frac{2i\upsilon}{\sqrt{\nu\omega}}\right) + \ldots, \quad h_{1,2}^0 = \pm i, \quad h_{3,4}^0 = \pm 1 \tag{2.7}$$

In Eq. (2.7) the coefficients of h^2 and h are assumed to be quantities of the same order of smallness for fairly large values of ω. If we assume that $c(\nu\omega)^{-1/2} \sim 1$ while $\upsilon(\nu\omega)^{-1/2} \ll 1$, the situation is similar to that considered above (relations (2.1)–(2.6)). The case (2.7) is considerably simpler for analytical investigation. Note that, in addition to the procedure of expansion in powers of the small quantity ν/ω, we can use the scheme of successive approximations, which is less lengthy for numerical calculations.

3. The problem of the natural vibration of a moving string

In the case of low flexural rigidity, when $\nu^2 \ll c^2$ (in the limit as $\nu \to 0$), problem (1.3), (1.4) is singularly degenerate. The equation and boundary conditions are modified in a natural way:

$$\ddot{u} + 2\upsilon\dot{u}' - (c^2 - \upsilon^2)u'' = 0, \quad u(t, 0) = u(t, 1) = 0 \tag{3.1}$$

A problem of the form (3.1) is often encountered in the literature (see Refs 1,2,8,9 and the bibliography). Using the approach described above, we will derive its complete analytical solution. For this purpose we will consider the complex Sturm–Liouville type problem[3] for determining natural frequencies and forms. Using relations (1.5) and (3.1) we obtain the problem (for $c^2 > \upsilon^2$)

$$U'' - 2i\alpha\Omega U' + \Omega^2 U = 0, \quad U(0) = U(1) = 0 \tag{3.2}$$

Boundary-value problem (3.2) is obtained from (3.1) by means of the replacement

$$u = U(x)\exp i\omega t, \quad \alpha = \frac{\beta}{\sqrt{1-\beta^2}}, \quad \beta = \frac{\upsilon}{c} < 1, \quad \Omega = \frac{\omega}{c\sqrt{1-\beta^2}}$$

It is not difficult to solve the non-self-conjugate problem (3.2). In fact, we obtain the following simple expressions for the roots $k_{1,2}(\Omega)$ of the intermediate equation of the type (1.6) and the forms $U(x)$

$$k^2 - 2i\alpha\Omega k + \Omega^2 = 0, \quad k_{1,2} = i\Omega(\alpha \pm \sqrt{1+\alpha^2})$$
$$U(x) = \frac{1}{2i}(c_1\exp k_1 x + c_2\exp k_2 x) = c_1\exp(i\alpha\Omega x)\sin(\sqrt{1+\alpha^2}\Omega x)$$
$$\alpha\Omega = \frac{\omega\upsilon}{c^2 - \upsilon^2}, \quad \sqrt{1+\alpha^2}\Omega = \frac{\omega c}{c^2 - \upsilon^2} \quad (c_2 = -c_1) \tag{3.3}$$

The coefficients c_1 and c_2 can be assumed to be real for simplicity (see below).

The non-self conjugate form of problem (3.2) can be proved by integration by parts. The equation that is conjugate to (3.2), as is easy to establish, contains the plus sign before the imaginary unit. The characteristic equation has a denumerable set of equidistant real roots Ω_n.

The natural frequencies ω_n can be found explicitly in an elementary way from condition $U(1) = 0$ (3.2), taking expression (3.3) for $U(x)$ into account:

$$\omega_n = \pi n c^{-1}(c^2 - \upsilon^2), \quad n = 1, 2, \ldots; \quad c = \left(\frac{TS}{\rho l^2}\right)^{1/2}, \quad \upsilon = \frac{\upsilon_0}{l} \tag{3.4}$$

Note that $\omega_n \to 0$ as $\upsilon \to c$, i.e., the natural vibrations of all modes disappear if $\upsilon^2 \geq c^2$.

The natural forms of vibrations of a moving string are complex. According to relations (3.3) and (3.4) they can be represented in the form (C_n are complex or real coefficients)

$$U_n(x) = C_n \exp(i\pi n c^{-1} \upsilon x)\sin(\pi n x), \quad n = 1, 2, \ldots \tag{3.5}$$

The property of non-orthogonality of the functions U_n and U_m, and also the functions U_n and the complex conjugate to it U_m^*, obviously follow from Eq. (3.5). A similar property holds for the real and imaginary parts of U_n and U_m.

The modified functions

$$X_n(x) = U_n(x)\exp(-i\pi n(c^{-1}\upsilon)x) = C_n \sin(\pi n x), \quad (X_n, X_m) = C_n C_m \delta_{nm}$$

will also be orthogonal, but it is doubtful if they can be effectively used as a basis.

We will now solve the initial problem (3.1). Substituting the values of the frequencies ω_n (3.4) and of the forms $U_n(x)$ (3.5) into expression (3.2) for the desired function $u(t, x)$, we obtain the eigenfunctions

$$u_n(t, x) = C_n \exp[i\pi n c^{-1}((c^2 - \upsilon^2)t + \upsilon x)]\sin(\pi n x), \quad n = 1, 2, \ldots \tag{3.6}$$

Since Eq. (3.1) is real (it does not contain complex coefficients), the real and imaginary parts of the function $u_f(t, x)$ (3.6) are real solutions; their linear combination with arbitrary real coefficients will also be a solution. This indicates that, in representation (3.6), the coefficients C_n can be arbitrary complex parameters of the form $C_n = A_n + iB_n = R_n \exp i\varphi_n$, where R_n is the modulus and φ_n is the phase, and the general real solution for the n-th mode is

$$u_n(t, x) = R_n \cos[\pi n c^{-1}((c^2 - \upsilon^2)t + \upsilon x) + \varphi_n]\sin(\pi n x)$$
$$R_n \ge 0, \quad 0 \le \varphi_n < 2\pi, \quad n = 1, 2, \ldots \tag{3.7}$$

Hence, if at a certain “initial” instant of time $t = t_0$ (=0) the distribution of the displacements and the velocities is specified in accordance with relations (3.7), then, when $t > 0$, the moving string will perform transverse wave motions of the n-th mode. The main important property is the fact that the vibrations are non-synchronous, i.e., the motions of the points do not have the form of standing waves, unlike the case of a fixed string.[6]

If it is required to take into account the weak flexural stiffness of the rod, due to elasticity, then, when $\nu^2 \ll c^2$ we can use the method of regular perturbations for the lower modes of vibration ($n = 1, 2, \ldots$) in the same way as in Section 2. An approximate solution of the auxiliary equation (1.6) for k can be constructed using the generating $k_{1,2}$ (3.3) in the form[5]

$$k_{1,2} = k_{1,2}^0 + \xi k_{1,2}^{(1)} + \ldots, \quad \xi = \frac{\nu^2}{c^2 - \upsilon^2} \ll 1$$
$$k_{1,2}^0 = i\Omega\,(\alpha \pm \sqrt{1+\alpha^2}), \quad k_{1,2}^{(1)} = \mp \frac{i\Omega^3(\alpha \pm \sqrt{1+\alpha^2})^4}{\sqrt{1+\alpha^4}}, \ldots \tag{3.8}$$

Hence it follows that the presence of an additional elastic stiffness leads to an increase in the natural frequencies, which corresponds to physical considerations. Additional terms of expansion (3.8) and the corresponding secular equation for ω, its roots $\Omega_n(\xi)$ and also the forms of the natural vibrations $U_n(x, \xi)$ and real solutions $u_n(t, x, \xi)$ for specified α are constructed in a standard way, as above.

4. Determination and analysis of the frequencie of natural vibration of a moving rod

The more complex limiting case of the wave motions of a slightly tensioned rod has not been investigated to any great extent; we will investigate these numerically-analytically. The flexural stiffness, leading to vibration is mostly accounted for by the elastic properties of the rod. Unlike the case of the vibration of a string considered in Section 3, the rate of displacement of the rod is not subject to limitations. We note the existence of approximate investigations of an applied problem by Bubnov's method based on a system of eigenfunctions, which are obtained when solving the problem for a fixed rod.[1,2] The numerical results obtained are in fact applicable for low vibration modes and low values of the velocity of motion ν_0 compared with the characteristic value $\sqrt{EI/(\rho l^2)}$. Their accuracy is essentially comparable with the accuracy of the analytical solution from Section 2, constructed above for a more general system (taking into account the tension for arbitrary values of $\sqrt{TS/\rho}$).

We will propose a precise numerical-analytical solution of the problem of the natural vibrations of a moving rod for an arbitrary mode of vibrations. To simplify the construction we will consider the case when the tension $T > 0$ (or the compression $T < 0$) can be neglected in the first approximation in terms of the parameter c/ν. In expressions (1.3), (1.5) and (1.6) we will assume that $c^2 = 0$, while the parameters $\nu > 0$ and υ can be arbitrary. Values of $c^2 > 0$ can then be taken into account by the perturbation method. In addition to the dimensionless argument x introduced above, we will introduce a dimensionless time τ and a dimensionless velocity α. Instead of boundary-value problem (1.3), (1.4) we now have

$$\ddot{u} + 2\alpha\dot{u}' + \alpha^2 u'' + u^{IV} = 0; \quad u(\tau, 0) = u'(\tau, 0) = u(\tau, 1) = u'(\tau, 1) = 0$$
$$u = u(\tau, x), \quad \tau = \nu t, \quad \alpha = \upsilon/\nu \tag{4.1}$$

The dot denotes a derivative with respect to τ. The initial conditions are not specified; it is required to determine the natural vibrations of the system, i.e., to calculate the frequencies and forms of these motions for arbitrary real values of the parameter α. For this purpose, we will make the standard change of variables of the type (1.5) in problem (4.1). We obtain a non-self-conjugate generalized boundary-value problem for eigenvalues and eigen functions[1,2,4]

$$\omega^2 U - 2i\alpha\omega U' - \alpha^2 U'' - U^{IV} = 0; \quad U(0) = U'(0) = U(1) = U'(1) = 0$$
$$u = U(x)\exp i\omega\tau \tag{4.2}$$

The solution is completed by determining the real values of the frequency ω_n and the family of complex forms $u_n(x)$. Expressions for the forms are then chosen from the condition for the required function $u_n(\tau, x)$ to be real: a fixed observer sees them. The non-self conjugate form of problem (4.2) is proved by integration by parts. The equation, conjugate to (4.2), will contain the minus sign before of the imaginary unit; coincidence (self-conjugacy) occurs when $\alpha = 0$ (see Sections 1–3).

At the initial stage we will investigate the problem of determining the natural frequencies ω. The corresponding characteristic equation is constructed in the same way as described above, based on the general solution of Eq. (4.2). For specified ω and α it will be sought in the form of an exponential function and reduced to a fourth-order equation of the type (1.6) (with $c^2 = 0$); it can be represented in the form

$$k^4 + \alpha^2 k^2 + 2i\alpha\omega k - \omega^2 = [k^2 - (-i\alpha k + \omega)][k^2 + (-i\alpha k + \omega)] = 0$$
$$U \sim e^{kx}; \quad k = k_j, \quad U_{(j)} = C_j \exp k_j x, \quad U_0 = \sum_j U_j; \quad j = 1, \ldots, 4 \tag{4.3}$$

According to Eq. (4.3) it is required to determine the complex roots of two complex second-order equations of the type (3.3). Assuming the parameter ω to be real, we obtain

$$k_{1,2} = -i\frac{\alpha}{2} \pm \begin{cases} \left(\omega - \dfrac{\alpha^2}{4}\right)^{1/2} \equiv -i\dfrac{\alpha}{2} \pm \gamma, & \omega \ge \dfrac{\alpha^2}{4} \\ i\left(\dfrac{\alpha^2}{4} - \omega\right)^{1/2} \equiv -i\dfrac{\alpha}{2} \pm i\gamma_*, & \omega < \dfrac{\alpha^2}{4} \end{cases}$$

$$k_{3,4} = i\frac{\alpha}{2} \pm i\left(\omega + \frac{\alpha^2}{4}\right)^{1/2} \equiv i\frac{\alpha}{2} \pm i\delta, \quad \omega > 0 \tag{4.4}$$

The quantities $k_{1,2}$ can be complex or pure imaginary, while $k_{3,4}$ can only be pure imaginary. The linearly independent solutions $U_j(x)$ have the form

$$U_{(1)} = \theta^-(x)\,\mathrm{sh}\,\gamma x, \ U_{(2)} = \theta^-\mathrm{ch}\,\gamma x, \ \omega \ge \frac{\alpha^2}{4}$$

$$U_{(1)} = \theta^-(x)\sin\gamma_* x, \ U_{(2)} = \theta^-(x)\cos\gamma_* x, \quad \omega < \frac{\alpha^2}{2}$$

$$U_{(3)} = \theta^+(x)\sin\delta x, \quad U_{(4)} = \theta^+(x)\cos\delta x; \quad \theta^{\pm} = \exp\left(\pm i\frac{\alpha}{2}x\right), \quad \omega > 0 \tag{4.5}$$

The general solution U_0 of the form (4.3) can be represented by linear combinations of the functions $U_{(j)}(x)$ with arbitrary complex coefficients C_j. From boundary conditions (4.2) and the requirements for the solution to be non-trivial we obtain the characteristic equation for the parameter ω in the form of two equations

$$\gamma\delta(\mathrm{ch}\gamma\cos\delta - \cos\alpha) - \frac{\alpha^2}{4}\mathrm{sh}\gamma\sin\delta = 0, \quad \omega \ge \frac{\alpha^2}{4} \tag{4.6}$$

$$\gamma_*\delta(\cos\gamma_*\cos\delta - \cos\alpha) - \frac{\alpha^2}{4}\sin\gamma_*\sin\delta = 0, \quad \omega < \frac{\alpha^2}{4} \tag{4.7}$$

The left-hand sides of these equalities are even functions of the parameter α, which corresponds to physical considerations.

As $\alpha \to 0$, Eq. (4.6) reduces to the well-known equation for the case of a fixed rod with clamped ends[7]

$$\mathrm{ch}\gamma\cos\gamma = 1, \quad \gamma = \delta = \sqrt{\omega}, \quad \alpha = 0$$
$$\omega_1 = 22.3733, \quad \omega_2 = 61.6728, \ldots, \quad \omega_n \approx (n + 1/2)^2\pi^2 + O(e^{-2\pi n}) \tag{4.8}$$

We will confine ourselves to a detailed numerical-analytical investigation of the lowest vibration modes, in particular $n = 1, 2$. Using the method of perturbations in powers of the small quantity $|\alpha|$, we establish that all the frequencies ω_n are even functions of α, i.e., they have

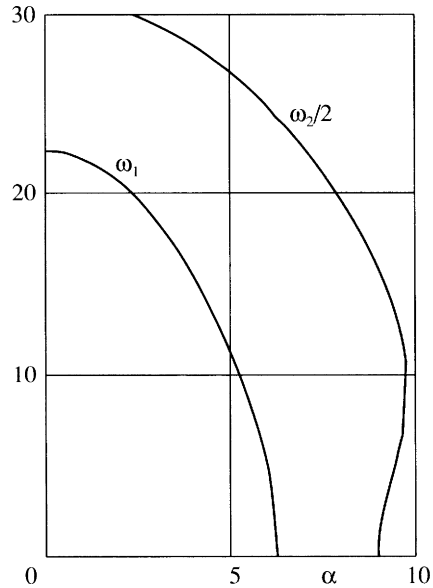

Fig. 2.

the following representations

$$\omega_n(\alpha) = \omega_n^0 + \alpha^2\omega_n^{(2)} + O(\alpha^4), \quad n = 1, 2, \dots$$
$$\omega_n^{(2)} = \frac{1}{4}\frac{1 + \text{th}\sqrt{\omega_n^0}\text{ctg}\sqrt{\omega_n^0}}{1 - \text{th}\sqrt{\omega_n^0}\text{ctg}\sqrt{\omega_n^0}} - \frac{2\sqrt{\omega_n^0} + (2\sqrt{\omega_n^0})^{-1}\text{sh}\sqrt{\omega_n^0}\sin\sqrt{\omega_n^0}}{\text{ch}\sqrt{\omega_n^0}\sin\sqrt{\omega_n^0} - \text{th}\sqrt{\omega_n^0}} < 0 \tag{4.9}$$

The coefficients $\omega_n^{(2)}$ are strictly negative, i.e., for small values of $|\alpha|$ the frequencies $\omega_n(\alpha)$ strictly decrease as $|\alpha|$ increases. Numerical solution of Eq. (4.6) for $n = 1, 2$ confirms that, as $|\alpha|$ increases, the quantity $\omega_{1,2}$ decreases so long as the inequality $\omega \geq \frac{\alpha^2}{4}$ is satisfied. When $\omega < \frac{\alpha^2}{4}$ the relation $\omega_{1,2}(\alpha)$ is determined numerically from Eq. (4.7). The corresponding graphs are shown in Fig. 2.

Critical values of the parameter α can be found numerically as the roots of the equation obtained from Eqs (4.6) and (4.7) when $\omega = \frac{\alpha^2}{4}$; we have the relations

$$\cos\frac{\alpha}{\sqrt{2}} - \cos\alpha = \frac{\alpha}{2\sqrt{2}}\sin\frac{\alpha}{\sqrt{2}}; \quad \alpha = \{\alpha_n^*\}$$
$$\alpha_1^* = 5.615, \quad \omega_1(\alpha_1^*) = 7.882; \quad \alpha_2^* = 9.68, \quad \omega_2(\alpha_2^*) = 22.43$$
$$\alpha_n^* \approx \sqrt{2}\pi n + \frac{4}{\sqrt{2}\pi n}(1 - (-1)^n\cos\sqrt{2}\pi n) + O(n^{-2}), \quad n \gg 1, \quad \omega_n^* = \frac{\alpha_n^{*2}}{4} \tag{4.10}$$

The quantities $\alpha_{1,2}^*$ and $\omega_{1,2}^*$ (4.10) are also obtained by solving Eqs (4.6) and (4.7). The values of the root $\omega_1(\alpha)$ for $\alpha \geq \alpha_1^*$ are found numerically by the method of continuation with respect to a parameter with corresponding monitoring of the accuracy. The calculations confirm that there is an unlimited increase in the derivative (the slope of the tangent) and lead to the need to reduce the variation of the parameter α and to increase the accuracy of the calculations. According to Eq. (4.9), the function $\omega_1(\alpha)$ decreases as α increases and vanishes when $\alpha_1^0 = 2\pi$; the tangent at this point is vertical (see Fig. 2). The first vibration mode disappears when $\alpha \geq \alpha_1^0$ and becomes impossible.

The behaviour of the function $\omega_2(\alpha)$ turns out to be much more complicated; when $\alpha > \alpha_2^*$ its extension is impossible: the second vibration mode disappears. At the point $\alpha = \alpha_2^*$, rotation of the lower part of the curve occurs (the tangent is vertical), which decreases as α decreases and vanishes when $\alpha_2^0 = 8.995 < \alpha_2^* = 9.68$ (the tangent is also vertical, see Fig. 2).

Hence, in the range $\alpha_2^0 < \alpha < \alpha_2^*$, the curve of $\omega_2(\alpha)$ is double-valued: vibrations with two frequencies-higher and lower – can exist in the system. Apparently, this effect has not been established in the literature. In a certain sense it is paradoxical. We would expect, physically, to obtain vibrations without sharp changes in mode, corresponding to the upper branch of the curve $\omega_2(\alpha)$, $0 \leq \alpha < \alpha_2^*$. The stability problem requires an additional investigation taking the non-linearity of the system into account. The difficulties of the analysis are increased by the fact that behaviour of the frequency curves $\omega_n(\alpha)$, $n \geq 3$ may turn out to be similar or even more complex (see below). It follows from the above analysis that when $\alpha > \alpha_2^*$, the second vibration mode does not exist.

The behaviour of the frequency curves $\omega_n(\alpha)$ in the region of values of α_n^0 for which $\omega_n(\alpha_n^0) = 0$ is of interest. Formally, when $\omega = 0$ Eq. (4.9) becomes an identity in α. It is required to take the accurate limit as $\omega \to 0$ in powers of ω, i.e., to represent the inverse function α in the form

$$\alpha(\omega) = \alpha^0 + \alpha^{(1)}\omega + \alpha^{(2)}\omega^2 + \dots \tag{4.11}$$

This expression for $\alpha(\omega)$ contains the unknowns $\alpha^0, \alpha^{(1)}, \alpha^{(2)}, \ldots$ which are determined by a standard method after substituting it into Eq. (4.7) and equating coefficients of like powers of ω. As has been already pointed out, when $\omega = 0$ it is satisfied identically. Further for unknown $\alpha^{(1)}$ the value $\alpha^{(1)} \equiv 0$ is obtained irrespective of the value of α^0. Finally, for $\alpha^{(2)}(\alpha^0)$ we obtain an expression which defines the required values of α^0

$$\alpha^{(2)} = -\frac{4}{\alpha^{0^4}}\frac{1-\cos\alpha^0}{\sin\alpha^0} + \frac{2}{\alpha^{0^3}}; \quad \{\alpha_n^0\} = \{\alpha_{2m-1}^0\} \cup \{\alpha_{2m}^0\}, \quad \alpha_{2m-1}^0 = 2\pi m,$$
$$\alpha_{2m}^0 = 2\chi_m, \quad m = 1, 2, \ldots$$
$$\chi_m = \mathcal{A}\mathrm{rg}_\chi(\mathrm{tg}\chi - \chi), \quad \chi_1 = 4.493, \quad \chi_2 = 7.725, \quad \chi_3 = 10.904, \ldots, \quad \chi_m \approx (m + 1/2)\pi \tag{4.12}$$

Thus, according to relations (4.10) and (4.12), the following distribution of the critical values of the velocity of the rod motion, normalized by the quantity ν (4.1), i.e., the values of the parameter α, has been obtained. For $\alpha = 0$ the natural frequencies $\omega_n(\alpha)$ are a maximum and are determined by the values of (4.8). The lowest frequency $\omega_1(\alpha)$ decreases to zero as α increases up to the limit value $\alpha_1^0 = 2\pi$ (4.12), since $\alpha_1^0 > \alpha_1^* = 5.615$ (see relations (4.10) and Fig. 2). The behaviour of the function $\omega_2(\alpha)$ is quite different, as described above, as a result of the satisfaction of the inverse inequality $\alpha_2^0 < \alpha_2^*$ according to the asymptotics (4.12) and (4.10) (see also Fig. 2). For the next vibration modes ($n \geq 3$) a comparison of the quantities α_n^0 and α_n^* also confirms that this inequality is satisfied. According to the asymptotics (4.12) and (4.10) we have

$$\alpha_n^0 \approx \pi n, \quad \alpha_n^* \approx \sqrt{2}\pi n, \quad \alpha_n^*/\alpha_n^0 = \sqrt{2}; \quad n \gg 1; \quad \omega_n(0) \approx (n + 1/2)^2\pi^2$$

Hence it follows that there is unlimited increase in the absolute difference between the critical velocities, corresponding to the interval of non-uniqueness in the dependence of ω_n on α, as the number of the mode n increases. The relative value of this interval approaches a constant value $(\sqrt{2} - 1)$.

Hence, we have carried out a fairly complete analysis of the spectral properties of a moving rod, as a function of the rate of displacement and the number of the mode, that is interesting from the mechanical point of view. New mechanical effects have been established which are of interest from the theoretical[3,4,6] and applied[1,2,7,8] points of view.

A comparison of the behaviour of the frequencies of natural vibrations of a moving rod and of a string (Sections 3 and 4) indicates considerable differences. According to Eqs (3.4) all the natural frequencies $\omega_n = 0$ when $\nu = c$, where c is the velocity of transverse waves of a fixed string. The dependence of its frequencies ω_n on ν is strictly convex upwards and all the derivatives are finite. The physical explanation of the differences is that there is no dispersion of transverse waves for a string: the phase and group velocities coincide. The transverse waves for a rod possess dispersion, the group velocity being higher than the phase velocity. For velocities of displacement ν as high as desired, there are fairly high vibration modes. A certain number of the lower vibration modes disappear in this case, according to relations (4.10).

5. Modelling of the natural vibration of a moving rod

We will now construct the natural forms of vibrations $U_n(x)$ based on the complex functions $U_{(j)}(x)$ (4.5) and the values ω_n of the natural frequencies obtained in Section 4. For brevity, the number of a mode (the subscript n) when deriving the required expressions is not indicated. We will obtain a "normalized" representation of the form

$$U^* = N^{-1}(V + iW), \quad N^2 = \int_0^1 [V^2(x) + W^2(x)]dx > 0 \tag{5.1}$$

The real part V and the imaginary part W of the "unnormalized" function U are found in terms of the complex expressions $U_{(j)}$, taking into account boundary conditions (4.2), which impose constraints on the coefficients $C_{(j)}$. The expressions obtained using a computer are extremely lengthy; for values of $\omega \geq \alpha^2/4$ we have

$$V = (B_{2sc}\theta_c - B_{2cs}\theta_s)\mathrm{sh}\gamma x + (B_{1sc}\theta_c - B_{1cs}\theta_s)\sin\delta x + (B_{3c}^-\theta_c + B_{3s}^+\theta_s)\mathrm{ch}\gamma x -$$
$$-(B_{3c}^-\theta_c - B_{3s}^+\theta_s)\cos\delta x$$
$$W = -(B_{2cs}\theta_c + B_{2sc}\theta_s)\mathrm{sh}\gamma x + (B_{1cs}\theta_c - B_{1sc}\theta_s)\sin\delta x + (B_{3s}^+\theta_c - B_{3c}^-\theta_s)\mathrm{ch}\gamma x -$$
$$-(B_{3s}^+\theta_c + B_{3c}^-\theta_s)\cos\delta x$$
$$\theta_s = \sin(\alpha x/2), \quad \theta_c = \cos(\alpha x/2), \quad \gamma = (\omega - \alpha^2/4)^{1/2}, \quad \delta = (\omega + \alpha^2/4)^{1/2} \tag{5.2}$$

The constant coefficients $B_{ksc}(k = 1, 2; s \leftrightarrow c)$, B_{3s}^+, B_{3c}^- are expressed explicitly in terms of the known parameters

$$B_{1sc} = -\alpha\theta_{s1}\mathrm{sh}\gamma + \gamma\theta_{c1}(\cos\delta - \mathrm{ch}\gamma), \quad B_{2sc} = -\alpha\theta_{s1}\sin\delta + \delta\theta_{c1}(\mathrm{ch}\gamma - \cos\delta)$$
$$B_{3s}^{\pm} = \theta_{s1}(\gamma\sin\delta + \delta\,\mathrm{sh}\gamma); \quad s \leftrightarrow c, \quad \theta_{s1} = \sin(\alpha/2), \quad \theta_{c1} = \cos(\alpha/2) \tag{5.3}$$

Similar standard calculations lead to representations for the functions V and W when $\omega < \alpha^2/4$. The changes compared with expressions (5.2) and (5.3) are due to the replacements

$$\gamma \to i\gamma_* = i(\alpha^2/4 - \omega)^{1/2}, \quad \text{sh}\gamma x = i\sin\gamma_* x, \quad \text{ch}\gamma x = \cos\gamma_* x \tag{5.4}$$

When $\alpha = 0$ the quantities γ and δ are identical $\gamma = \delta = \sqrt{\omega}$, but the second possibility for $k_{1,2}$ is not realized (see problem (4.2)); in this case $W(x) \equiv 0$, and the functions $v(x)$ are identical with the forms of the natural vibrations of a fixed rod with clamped ends. Note that, as in Section 3 for the natural vibrations of a string, the form U is determined, apart from a complex factor $R\exp i\,\Phi$, where $R > 0$, $0 \le \Phi < 2\pi$. Hence, for the required real solution of the initial problem (4.1), based on the "normalized" functions U^* (5.1), we obtain the expression

$$u(\tau, x) = \text{Re}[R\exp(i(\omega\tau + \Psi))U^*(x)] = RN^{-1}[V\cos(\omega\tau + \Psi) - W\sin(\omega\tau + \Psi)] =$$
$$= r\cos(\theta + \varphi)$$
$$r = RN^{-1}(V^2 + W^2)^{1/2}, \quad \varphi = \Phi + \Psi, \quad \text{tg}\Psi = \frac{W}{V}, \quad \theta = \omega\tau \tag{5.5}$$

which depends on two arbitrary real parameters. Note that for different vibration modes $u_n(\tau, x)$ one must substitute into expression (5.5) the corresponding quantities R_n, φ_n, N_n, ω_n and the functions $V_n(x)$ and $W_n(x)$, defined by relations (5.1)–(5.4) for the coefficients $B_{ksc}(k = 1, 2; s \leftrightarrow c)$, B^+_{3s}, B^-_{3c}, known for the mode n considered. The functions $u = u_n(\tau, x)$ (5.5) for $R = 1$ and $\Phi = 0$ for $n = 1, 2$ and the required admissible values of the parameter α, give a clear representation of the natural vibrations of a moving rod.

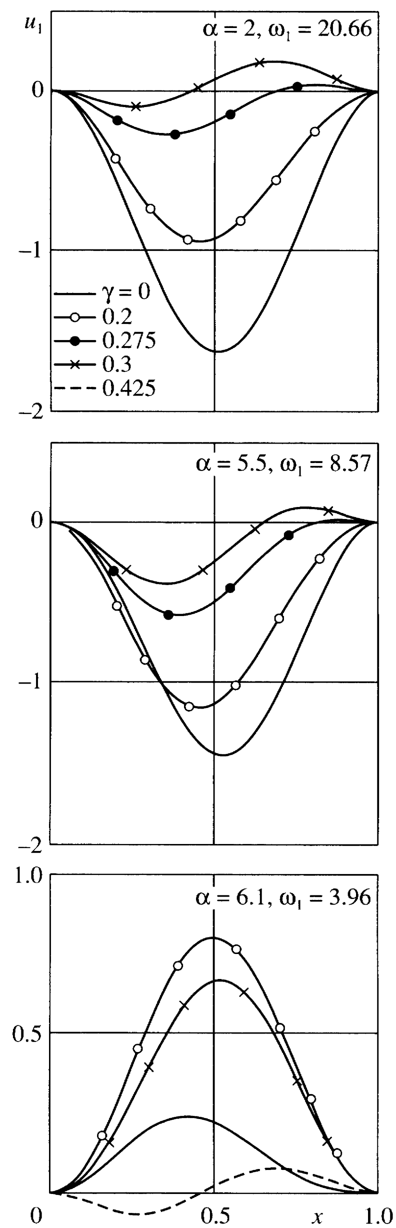

Fig. 3.

We will derive and comment on some results of mathematical modelling, which illustrate the unusual properties of the natural vibrations of a rod for relatively small, medium and large values of the displacement velocity, i.e., the values of the parameter α. For a clear representation of the vibration forms of a moving rod from the position of a fixed observer, formulae (5.1)–(5.5) are fed into the computer and animated.

The main property of the motion consists in a radical difference from the generally known pattern of standing waves with fixed nodal points and synchronism of the vibrations, present in the self-conjugate problems (for $\alpha = 0$).[3,6] The wave motions of the rod are extremely exotic: "quasinodal" points are periodically displaced along the x axis, and their number also changes periodically. The rod does not become rectilinear at the instant of time $\tau \in \left[0, \frac{2\pi}{\omega}\right]$ since the phase correction ψ (5.6) depends considerably on the x coordinate (the functions $V(x)$ and $W(x)$ are not proportional). The number of quasinodal points may not correspond to the mode number, i.e., it may be greater or smaller ($n \geq 2$).

We will consider the first and second vibration modes for different values of the parameter α over half a period $0 \leq \tau \leq \frac{T_{1,2}}{2} = \frac{\pi}{\omega_{1,2}}$. The phases $\theta_{1,2} = \omega_{1,2}\tau$ vary in the range $0 \leq \theta_{1,2} \leq 2\pi$; we will take their values $\theta_{1,2} = 2\pi\gamma$ for different values of γ. It is sufficient to draw graphs for $0 \leq \gamma < 1/2$, since they are antisymmetric about the middle of the rod $x = ½$ and values of the phase $\theta_{1,2} = \pi$, i.e., $\gamma = ½$.

Curves of u_1 for different values of the phase θ_1 and $\alpha = 2$, $\omega_1 = 20.66$ (i.e., comparatively small values of α), $\alpha = 5.5$, $\omega_1 = 8.57$ (close to the critical value $\alpha_1^* = 5.615$ (4.10), see Fig. 2), and $\alpha = 6.1$, $\omega = 3.96$ (i.e., α close to the critical value $\alpha_1^0 = 2\pi$ (4.12), see Fig. 2) are presented in Fig. 3. It follows from the first family of curves in Fig. 3 that when $\gamma \approx ½$ (the quarter of a period) a typical single-mode form ($\gamma = 0$) becomes a "two-mode form: in the vicinity of $x = 1$ the curve becomes convex and an additional node (zero) appears, which is inherent in the second vibration mode. Points near the right end begin to move faster than points located to the left; the zero is shifted to the left (the curves for

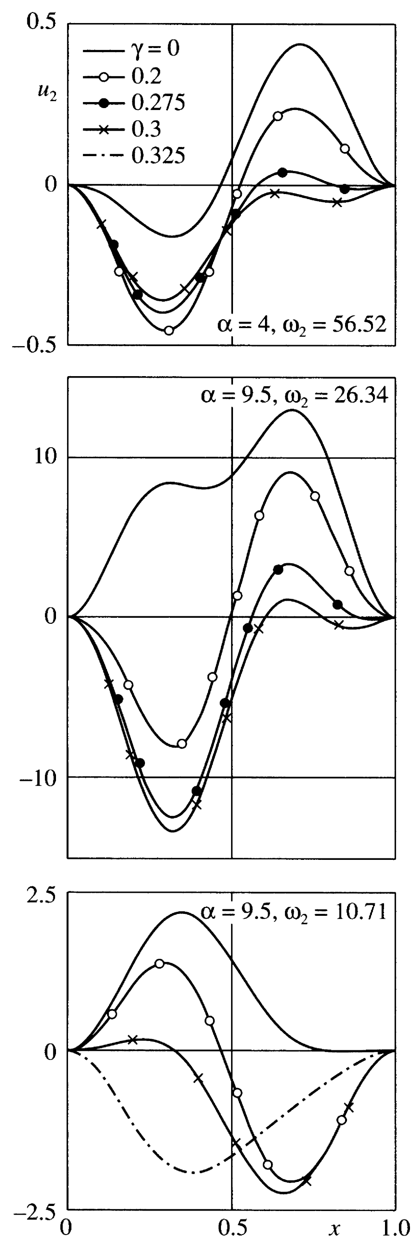

Fig. 4.

$\gamma = 0.275$ and $\gamma = 0.3$). As $\gamma \to 1/2$ the curve becomes convex and antisymmetric with respect to $x = 1/2$ compared with the initial curve (with $\gamma = 0$), i.e., close to the usual single-mode form. When $\gamma \approx 3/4$ an additional node, moving to the right, again appears close to the left end ($x = 0$) of the moving rod. Points of the rod close to the left end when moving downwards begin to move faster than points located on the right, and the node is displaced to the right. When $\gamma \to 1$ the rod becomes convex downwards, as for $\gamma = 0$.

In the subcritical range of values of the parameter α (the velocity of motion v) the vibration pattern is qualitatively close to that described above. For $\alpha = 5.5$ (the middle family of curves in Fig. 3) it can be seen that the nodal point for $\gamma \geq 1/4$ moves somewhat more slowly. This can be explained by considerably less effective stiffness or more effective inertia of the elements of the more rapidly moving rod. The natural frequency is approximately 2.5 lower less than for $\alpha = 2$.

The supercritical region of values of the parameter $\alpha = 6.1(\alpha_1^* < \alpha < \alpha_1^0)$ leads to a low frequency and a considerable change in the vibration pattern (the lower family of curves in Fig. 3). Over a considerable part of the half-period, i.e., in the range $0 \leq \gamma \leq 0.425$, the curves do not have additional nodes and correspond qualitatively to the first mode, but the values and positions of the maxima are changed considerably. Over a comparatively narrower range $0.425 \leq \gamma \leq 0.475$ the curve sags, and the node is displaced from left to right. When $\gamma = 1/2$ the curve is antisymmetric about $x = 1/2$ to the curve for $\gamma = 0$. Further, when $1/2 \leq \gamma \leq 1$ the shape of the curves evolves antisymmetrically.

We will consider three families of curves of u_2, corresponding to the second vibration mode. For a moderately large value of the parameter $\alpha = 4$, $\omega_2 = 56.52$ (Fig. 4, the upper family of curves) at the initial stage, the pattern of natural forms corresponds to the standard one: in the region of the middle of the interval $x \approx 1/2$ there is a single node, which is shifted to the right as γ increases (the curve for $\gamma = 0.275$). Points of the rod close to the right end rapidly sink downwards, the curve sags, and for $\gamma \approx 1/4$ an additional node appears, characteristic of the third vibration mode (the curves for $\gamma = 0.275$ and $\gamma = 0.3$). A comparatively rapid motion of the elements close to the right end leads to merging of both nodes, and when γ increases further they disappear, which corresponds to the first mode: the curve has an odd "double-humped" shape (with three extrema). A downward bulge then occurs at the right end and a relatively rapid rise in the elements of the rod close to its left end ($x \leq 1/2$). When $\gamma = 1/2$ the curve becomes antisymmetric with respect to the curve for $\gamma = 0$. The process continues further antisymmetrically and for $\gamma = 1$ the curve returns to the initial state, corresponding to $\gamma = 0$.

We will present a family of curves for the values $\alpha_2^* < \alpha < \alpha_2^0$, close to critical (see formulae (4.10), (3.12) and Fig. 2), to which two natural frequencies correspond (one higher and one lower). We will take $\alpha = 9.5$ and the higher value $\omega_2 = 36.34$ (the central family of curves in Fig. 4). The family of curves corresponds qualitatively to the case $\alpha = 4$, but with a shift in time due to the change in the phase θ (5.5). Forms that are close to the standard two-mode form for a fixed rod ($\gamma = 0.2$), asymmetric deformation and the occurrence of second quasinodes ($\gamma = 0.275$ and $\gamma = 0.3$) are observed. When $\gamma = 0$ the curve has no nodes and is rather fanciful, being similar in form to the curve of $\gamma = 0.3$ for the upper family, shown in Fig. 4.

In conclusion we present the family of curves (Fig. 4) for the same value of $\alpha = 9.5$ and the lower frequency $\omega_2 = 10.71$. For $\gamma = 0$ the form of the curve corresponds to the first vibration mode with an extremely high degree of "attachment" at the right end ($x \leq 1$). An increase in the phase ($\gamma = 0.2$, $\gamma = 0.3$) leads to a sharp and considerable sag of the curve; a single node occurs in this case which moves rapidly from right to left. For $\gamma \geq 0.31$ the node disappears (it merges with the point $x = 0$) and the curve takes a single-mode form ($\gamma = 0.325$). A relatively slow deformation of the curve then occurs to a form that is asymmetrical to the curve $\gamma = 0$ with the above-mentioned "attachment" of the left part of the rod ($x \geq 0$). The pattern of the vibrations corresponds qualitatively to that investigated above for the first vibration mode (Fig. 3). Hence, the forms of the vibrations at low frequency differ considerably from the form u for high frequencies (the velocity of displacement is the same).

The nature of the vibrations for the subsequent modes ($n \geq 3$) is similar to the case $n = 2$.

Thus, we have established and investigated previously unknown forms of the vibrational motions of elastic systems. The wave motions are described by non-self conjugate boundary-value problems, and hence the use of Bubnov–Ritz methods leads to fundamental difficulties. The results obtained are of both theoretical and practical value in machine building, metallurgy, tethered systems, transport pipelines, the textile industry and other areas of technology.

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